MATH1013 Calculus IB
Final Examination
(December 12, 2024; 12:30 – 15:30)

Name: ____
Student ID Number: ____
Lecture Section: ____

Directions:

- This is a closed-book examination. **Calculator of any kind is NOT allowed in this exam**.
- **DO NOT use any of your own scratch paper**. Use only the scratch paper provided by the examination. DO NOT take away any scratch paper away from the exam venue.
- Write your name, student ID number and lecture section in the space provided above and also in the **Multiple Choice Answer Sheet** provided. Also **mark your student ID number using the bubbles** in the Multiple Choice Answer Sheet using an HB pencil.

The HKUST Academic Honor Code

Honesty and integrity are central to the academic work of HKUST.

Students of the University must observe and uphold the highest standards of academic integrity and honesty in all the work they do throughout their program of study.

As members of the University community, students have the responsibility to help maintain the academic reputation of HKUST in its academic endeavors.

Please read the following statement and sign below it.

I confirm that I have answered the questions using only materials specifically approved for use in this examination, that all the answers are my own work, and that I have not received any assistance during the examination.
I understand that sanctions will be imposed, if I am found to have violated the University's regulations governing academic integrity and honesty.

Student's Signature:

____

Question No.	Score	Out of
Q1 – 21		51
Q22		13
Q23		11
Q24		12
Q25		13
Total score		**100**

Note:

- The full score of this paper is 100.
- Recommended time allocation:

Question No.	Time
Q1 – 21	80 minutes
Q22	20 minutes
Q23	20 minutes
Q24	20 minutes
Q25	25 minutes

- Unless stated otherwise, you are allowed to use the following formulas in your work without proof:

Let x, y *and* r *be real numbers with* $r \neq 1$, *and* n *be a positive integer. Then*

$$\cos^2 x + \sin^2 x = 1$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\sin x \pm \sin y = 2\sin\frac{x \pm y}{2}\cos\frac{x \mp y}{2}$$
$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$
$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$2\sin x\cos y = \sin(x+y) + \sin(x-y)$$
$$2\cos x\sin y = \sin(x+y) - \sin(x-y)$$
$$2\cos x\cos y = \cos(x+y) + \cos(x-y)$$
$$2\sin x\sin y = \cos(x-y) - \cos(x+y)$$

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=0}^{n-1} r^k = \frac{r^n - 1}{r - 1}$$

Part A: Multiple choice problems (51 points)

For each of the following problems, choose **one and only one** correct option. The problem Q1 is worth 1 point, and each of the problems Q2 – Q21 is worth 2.5 points.

Please mark your answers to the multiple choice problems in the Multiple Choice Answer Sheet.

1. **Choose option A for this problem**. This is for the grading machine to classify the correct version of your exam paper.

   A. **Please choose this option.**
   B.
   C.
   D.
   E.

2. Let a and b be real numbers and let $f: \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} x^2 + ax + b & \text{if } x \geq 0 \\ \dfrac{\sin x}{x} + 2x & \text{if } x < 0 \end{cases}.$$

   If f is **differentiable** at 0, what is the value of a?

   A. $a = -\frac{1}{2}$.
   B. $a = 0$.
   C. $a = 1$.
   D. $a = \frac{3}{2}$.
   E. $a = 2$.

3. Evaluate the integral

$$\int_0^{\frac{\pi}{4}} \sec^2 x \tan^2 x \, dx.$$

A. $\frac{1}{3}$.

B. $1 - \frac{\pi}{4}$.

C. $\frac{2\sqrt{2}-1}{3}$.

D. $\frac{2\sqrt{2}}{3}$.

E. 2.

4. Evaluate the limit

$$\lim_{x \to 0} \frac{e^{2x} - e^{2\sin x}}{x^3}.$$

A. $\frac{1}{2}$.

B. $\frac{1}{3}$.

C. $\frac{1}{6}$.

D. $\frac{1}{12}$.

E. The limit does not exist.

5. To which of the following equations can we find an approximated real number solution using **Newton's method** with the initial trial $x_0 = 1$?

(i) $\cos x = x$

(ii) $e^{x-1} = x + 1$

(iii) $x \ln x = x + 2$

A. (i) only.
B. (i) and (ii) only.
C. (i) and (iii) only.
D. (ii) and (iii) only.
E. (i), (ii) and (iii).

6. Let $a < c < b$ be real numbers and let $f: (a, b) \to \mathbb{R}$ be a continuous function. If

$$f'(x)\begin{cases} > 0 & \text{for } a < x < c \\ \text{is undefined} & \text{for } x = c \\ > 0 & \text{for } c < x < b \end{cases} \quad \text{and} \quad f''(x)\begin{cases} > 0 & \text{for } a < x < c \\ \text{is undefined} & \text{for } x = c \\ < 0 & \text{for } c < x < b \end{cases},$$

which of the following is a correct description of the point with coordinates $(c, f(c))$?

A. It is a local maximum point of f.
B. It is a local minimum point of f.
C. It is a point of inflection of f.
D. It is not a point on the graph of f.
E. None of the previous is correct.

7. Which of the following functions F is/are an **antiderivative** of

$$f(x) = \sec x$$

on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$?

(i) $F(x) = \ln(\sec x + \tan x)$

(ii) $F(x) = \ln\left(\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right)$

(iii) $F(x) = \frac{1}{2}\ln\left(\frac{1 + \sin x}{1 - \sin x}\right)$

A. None of them.
B. (i) only.
C. (i) and (ii) only.
D. (i) and (iii) only.
E. (i), (ii) and (iii).

8. Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function that is <u>**non-differentiable**</u> at 0. Let $g: \mathbb{R} \to \mathbb{R}$ be the function

$$g(x) = xf(x).$$

Compute $g'(0)$ if it exists.

A. $g'(0) = 0$.
B. $g'(0) = f(0)$.
C. $g'(0) = f'(0)$.
D. $g'(0) = f(0) + f'(0)$.
E. $g'(0)$ does not exist.

9. Let $f, g: [a, b] \to \mathbb{R}$ be continuous functions, both differentiable on (a, b), such that

$$f(x) \leq g(x) \qquad \text{for every } x \in [a, b].$$

Which of the following statements **<u>may not be true</u>**?

(i) For every $c \in (a, b)$, we always have

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

(ii) For every $x \in (a, b)$, we always have

$$f'(x) \leq g'(x).$$

(iii) For every n numbers $x_0, x_1, \dots, x_n \in [a, b]$, we always have

$$\sum_{k=0}^{n} f(x_k) \leq \sum_{k=0}^{n} g(x_k).$$

(iv) We always have

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

A. Only (i) may not be true.
B. Only (ii) may not be true.
C. Only (iii) may not be true.
D. Only (iv) may not be true.
E. All the four given statements must be true.

10. Consider the following pairs of functions:

(i) $f(x) = 1 + e^{-x}$ and $g(x) = e^{-2x}$;
(ii) $f(x) = 3x + \cos x$ and $g(x) = 4x - \sin x$;
(iii) $f(x) = \ln 5x$ and $g(x) = 6^x$.

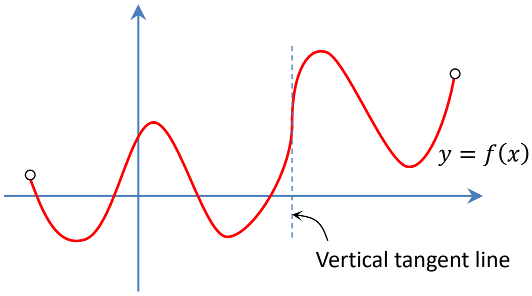
For which of these pairs of functions can we conclude that

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} \frac{f'(x)}{g'(x)}$$

<u>directly using l'Hôpital's rule</u>?

A. None of them.
B. (i) only.
C. (iii) only.
D. (ii) and (iii) only.
E. (i), (ii) and (iii).

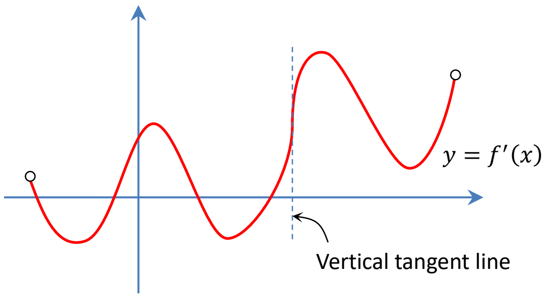
11. Let f be a function whose graph is as shown below.



How many local maximum point(s) does **its derivative** f' have?

A. 5.
B. 4.
C. 3.
D. 2.
E. 1.

12. Let f be a function **whose derivative** f' has its graph as shown below.



How many points of inflection does the function f have?

A. 2.
B. 3.
C. 4.
D. 5.
E. 6.

13. Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function. Which of the following statements are true?

(i) If $\lim_{x \to +\infty} f'(x) = 0$, then $\lim_{x \to +\infty} f(x)$ exists as a real number.

(ii) If $\lim_{x \to +\infty} f(x)$ exists as a real number, then $\lim_{x \to +\infty} f'(x) = 0$.

(iii) If $\lim_{x \to +\infty} f'(x) = m \neq 0$, then f has a slant asymptote.

A. None of them.
B. (i) only.
C. (ii) only.
D. (i) and (iii) only.
E. (i), (ii) and (iii).

14. Which one of the following theorems should be applied in order to conclude that the solution to the equation

$$\cos x = x$$

in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is **unique** (i.e. there is **no more than one** solution in the interval)?

A. Intermediate Value Theorem.
B. Squeeze Theorem.
C. Newton's method.
D. Rolle's Theorem.
E. Fundamental Theorem of Calculus.

15. Let a and b be a pair of real numbers. Which of the following inequalities **must** hold?

(i) $|\sin a - \sin b| \leq |a - b|$.
(ii) $|\tan a - \tan b| \geq |a - b|$.
(iii) $|\sec a - \sec b| \geq |a - b|$.

A. (i) only.
B. (i) and (ii) only.
C. (i) and (iii) only.
D. (ii) and (iii) only.
E. (i), (ii) and (iii).

16. Let $f: [0, +\infty) \to \mathbb{R}$ be the function

$$f(x) = \int_0^x e^{\sqrt{xt}} dt\,.$$

Compute the **derivative** of f, i.e. compute $f'(x)$ for $x > 0$.

A. $f'(x) = e^{\sqrt{x}}$.
B. $f'(x) = e^x$.
C. $f'(x) = 2xe^x$.
D. $f'(x) = \frac{1}{x} e^{\sqrt{x}} - \frac{1}{x} f(x)$.
E. $f'(x) = 2e^x - \frac{1}{x} f(x)$.

17. Evaluate the integral

$$\int_0^2 (1 + 3x)\sqrt{4 - x^2}\, dx.$$

A. $\frac{\pi}{2} - 8$.
B. $\frac{\pi}{2} + 8$.
C. 8π.
D. $\pi - 8$.
E. $\pi + 8$.

18. Evaluate the limit

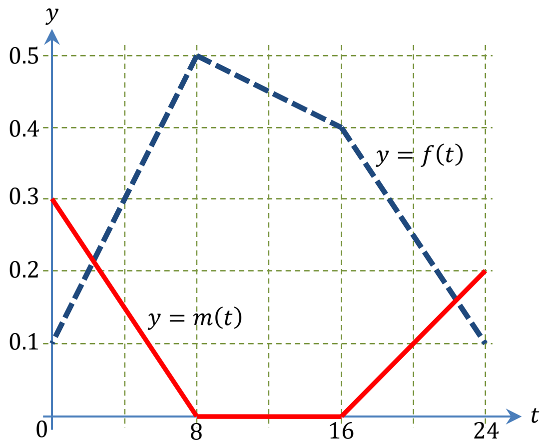
$$\lim_{n \to +\infty} \left(\frac{1}{n} \cos \frac{\pi}{2n} + \frac{1}{n} \cos \frac{2\pi}{2n} + \frac{1}{n} \cos \frac{3\pi}{2n} + \cdots + \frac{1}{n} \cos \frac{(n-1)\pi}{2n} + \frac{1}{n} \cos \frac{n\pi}{2n} \right).$$

A. 1.
B. $\sin 1$.
C. $\frac{2}{\pi}$.
D. $\frac{2}{\pi} \sin 1$.
E. 0.

19. Let $f, m: [0, 24] \to \mathbb{R}$ be functions whose graphs are as shown in the diagram below.

- ⊙ $f(t)$ (dashed graph, inches per hour) represents the **rate of snowfall** and
- ⊙ $m(t)$ (solid graph, inches per hour) represents the **rate of snow melt**

at a location at time t (in hours).

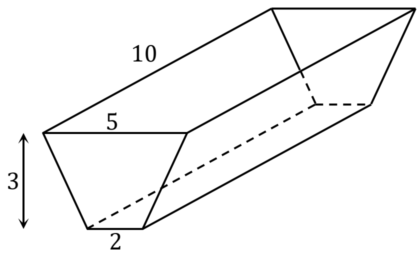


If there was 15 inches of snow on the ground at $t = 0$ hours, how much snow was there on the ground at $t = 24$ hours?

A. 30 inches.
B. 23 inches.
C. 21 inches.
D. 13 inches.
E. 6 inches.

20. A water trough is $10\ \text{m}$ long and its cross-section has the shape of an isosceles trapezium that is $2\ \text{m}$ wide at the bottom, $5\ \text{m}$ wide at the top, and has height $3\ \text{m}$. If the trough is being filled with water at a rate of $4\ \text{m}^3$ per minute, how fast is the water level rising when the depth of water is $2\ \text{m}$?

A. $\frac{1}{40}$ m per minute.
B. $\frac{1}{10}$ m per minute.
C. $\frac{1}{5}$ m per minute.
D. 1 m per minute.
E. None of the previous.



21. A particle is initially at rest. It then travels along a straight line, and its acceleration $a(t)$ (in $\mathrm{m/s^2}$) after t seconds is always given by

$$a(t) = 2e^{-t}.$$

How far is the particle from its initial position at the time when its velocity is $1\ \mathrm{m/s}$?

A. $\frac{1}{2}$ m.

B. $\frac{2}{e}$ m.

C. $\frac{1}{2}(\ln 2)^2$ m.

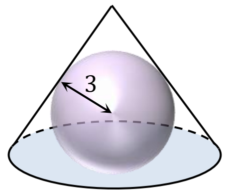
D. $(2\ln 2 - 1)$ m.

E. $\frac{1}{2}(\ln 2)(2 - \ln 2)$ m.

Part B: Short written problems (49 points)

Write down **detailed solutions** to each of the following problems Q22 – Q25. The back side of each page can be used for scratch work. **Indicate clearly on the front side** if you need to use the back side for answers.

22. Consider a fixed sphere of radius 3 units and a circular cone that circumscribes the sphere, as illustrated in the diagram on the right.



(a) What is the smallest possible **volume** of the circular cone? Give complete justification of your work in order to receive full credit.

(8 points)

Hint: If a circular cone has base radius r and height h, then its **volume** is

$$V = \frac{1}{3}\pi r^2 h.$$

(b) Show that the **<u>total</u> surface area** of the circular cone is also the smallest when its volume attains minimum as in (a). Also compute this smallest total surface area.

(5 points)

Hint: If a circular cone has base radius r and slant height l, then its **curved surface area** is

$$A_{\text{curved}} = \pi r l.$$

23. In the grid provided below, sketch the graph of a function f which satisfies all the following conditions:

(i) The domain of f is $\mathbb{R}$.

(ii) f is continuous on $(0, +\infty)$.

(iii) f is an odd function.

(iv) $f'(x) \begin{cases} > 0 & \text{if } 0 < x < 2 \text{ or } x > 4 \\ < 0 & \text{if } 2 < x < 4 \end{cases}$.

(v) $f''(x) \begin{cases} > 0 & \text{if } 0 < x < 1 \\ < 0 & \text{if } 1 < x < 4 \end{cases}$.

(vi) $f''(x) = 0$ if $x > 4$.

(vii) $\displaystyle\lim_{x \to 0^+} \frac{f(x) - 1}{x} = 0$.

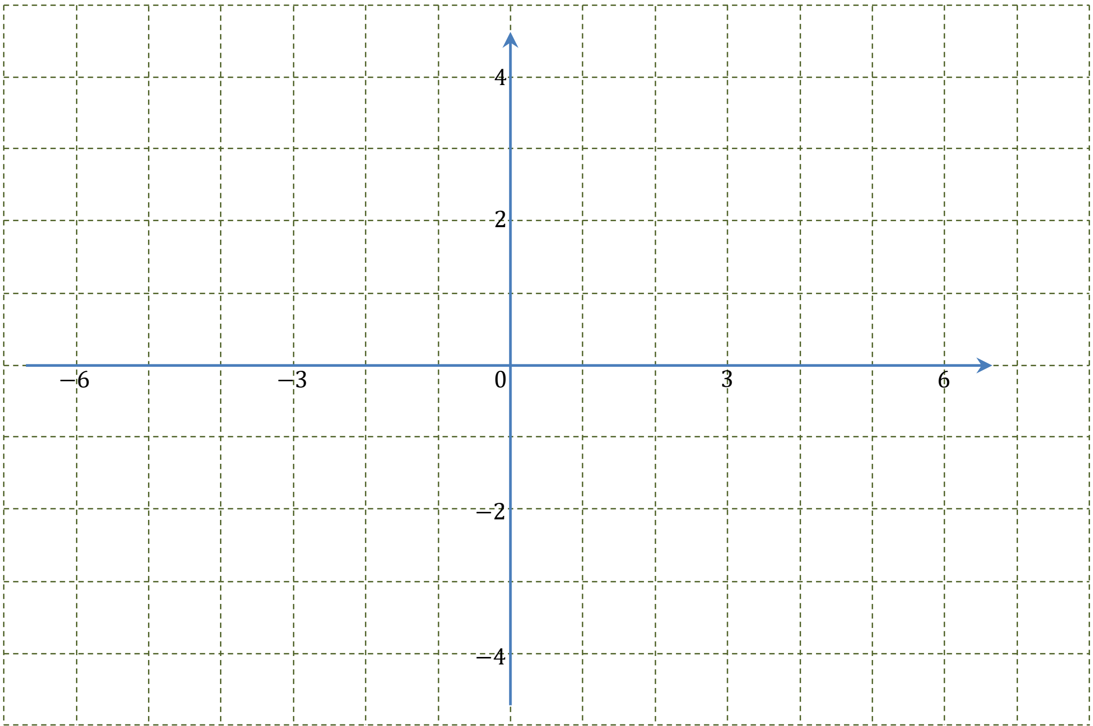
(viii) $\displaystyle\lim_{x \to 4^-} \frac{f(x) - f(4)}{x - 4} = -\infty$ and $\displaystyle\lim_{x \to 4^+} \frac{f(x) - f(4)}{x - 4} = 1$.

Please also annotate your graph by

- ⦿ <u>writing “**max**”</u> next to every local maximum point,
- ⦿ <u>writing “**min**”</u> next to every local minimum point and
- ⦿ <u>writing “**infl**”</u> next to every point of inflection of the graph of f in your sketch.

You are suggested to **<u>sketch with a pencil</u>** for easy amendment.

(11 points)



24. Let $f, g: \mathbb{R} \to \mathbb{R}$ be functions whose derivatives f' and g' are both continuous on $\mathbb{R}$. The following table shows some values of the functions f, g, f' and g'.

x	-2	0	1	3
$f(x)$	1	-2	2	1
$g(x)$	3	2	4	0
$f'(x)$	2	1	-1	-3
$g'(x)$	-1	3	0	-4

(a) For each of the following two statements, determine whether it is true or false by circling your answer. Justify your choice with either a proof or a counter-example.

(6 points)

(i) There exists a number $a \in (-2, 0)$ such that $f(a) = 0$.

Circle your answer: **TRUE** **FALSE**

Justification:

(ii) There exists a number $b \in (-2, 0)$ such that $f'(b) = 0$.

Circle your answer: **TRUE** **FALSE**

Justification:

Copied from the previous page:

Let $f, g: \mathbb{R} \to \mathbb{R}$ be functions whose derivatives f' and g' are both continuous on $\mathbb{R}$. The following table shows some values of the functions f, g, f' and g'.

x	-2	0	1	3
$f(x)$	1	-2	2	1
$g(x)$	3	2	4	0
$f'(x)$	2	1	-1	-3
$g'(x)$	-1	3	0	-4

(b) (i) Evaluate the integral

$$\int_0^3 f(x-2)f'(x-2)dx.$$

(3 points)

(ii) Evaluate the limit

$$\lim_{x\to 3}\frac{1}{x-3}\int_1^{f(x)} g(t)dt\,.$$

(3 points)

25. (a) By considering the global minimum (i.e. absolute minimum) of the function

$$f(x) = e^x - x,$$

prove that the inequality

$$e^x \geq 1 + x$$

holds for every real number x. (3 points)

(b) Using the result from (a), deduce that if $x \in (-1, 1)$, then

$$\ln(1+x) \leq x \leq -\ln(1-x).$$

(2 points)

(c) Let n be a positive integer. Using the result from (b), deduce that

$$\ln(2n+1) - \ln(n+1) \leq \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+n} \leq \ln(2n) - \ln n.$$

(3 points)

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(c) Let n be a positive integer. Deduce that

$$\ln(2n+1)-\ln(n+1) \leq \frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{n+n} \leq \ln(2n)-\ln n.$$

(d) Using the result from (c), compute the integral

$$\int_1^2 \frac{1}{x}\,dx$$

from the Riemann sum definition, **<u>without using Fundamental Theorem of Calculus</u>**.

(5 points)

END OF PAPER